Recall that for a periodic function with period, T, the function can be expressed as an infinite sum of sines and cosines (i.e. Fourier Series).

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T}t\right) + b_n \sin\left(\frac{2\pi n}{T}t\right) \quad (1)$$

where:

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt \quad (2)$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi n}{T}t\right)dt \quad (3)$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi n}{T}t\right)dt \quad (4)$$

Consider the case where $f(t)$ is a PWM waveform as shown in figure 1.

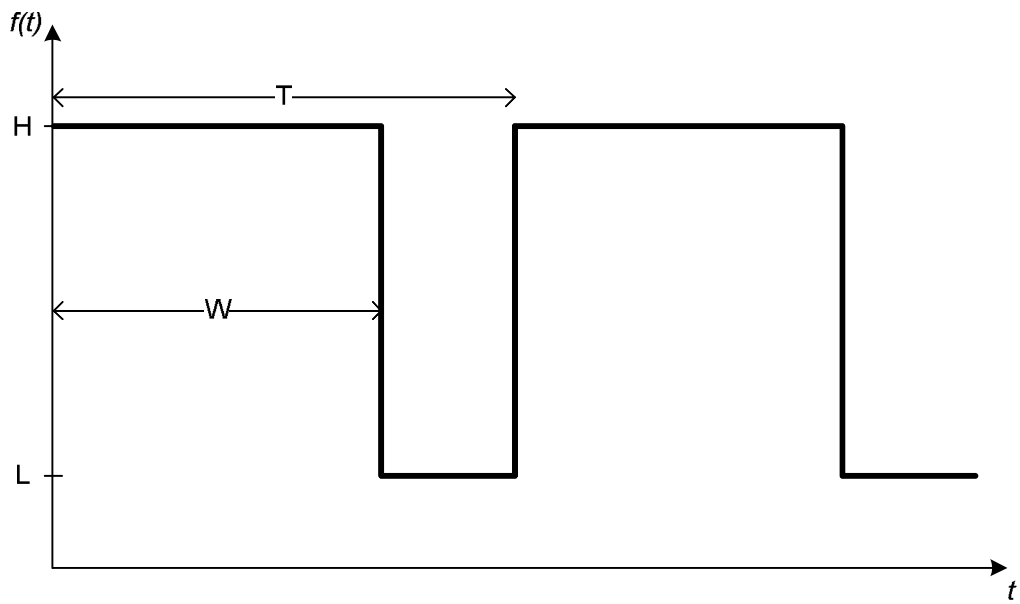

Figure 1. PWM function

$$f(t) = \begin{cases} H \;; & 0 \le t \le W \\ L \;; & W \le t \le T \end{cases} \quad (5)$$

So we have:

$$a_0 = \frac{1}{T}\left[\int_0^W H dt + \int_W^T L dt\right] = \frac{1}{T}\left[HW + L(T-W)\right] = H \cdot DC + L(1-DC)$$

$$
\begin{aligned}
a_n &= \frac{2}{T}\left[\int_0^W H\cos\left(\frac{2\pi n}{T}t\right)dt + \int_W^T L\cos\left(\frac{2\pi n}{T}t\right)dt\right] \\
&= \frac{2}{T}\left[H\frac{T}{2\pi n}\sin\left(\frac{2\pi n}{T}t\right)\Big|_0^W + L\frac{T}{2\pi n}\sin\left(\frac{2\pi n}{T}t\right)\Big|_W^T\right] \\
&= \frac{2}{T}\frac{T}{2\pi n}\left\{H\left[\sin\left(\frac{2\pi n}{T}W\right)-\sin(0)\right]+L\left[\sin(2\pi n)-\sin\left(\frac{2\pi n}{T}W\right)\right]\right\} \\
&= \frac{1}{\pi n}\left[H\sin\left(\frac{2\pi n}{T}W\right)-L\sin\left(\frac{2\pi n}{T}W\right)\right] \\
&= \frac{H-L}{\pi n}\sin\left(\frac{2\pi n}{T}W\right) \\
&= \frac{H-L}{\pi n}\sin(2\pi\cdot DC\cdot n)
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{2}{T}\left[\int_0^W H\sin\left(\frac{2\pi n}{T}t\right)dt + \int_W^T L\sin\left(\frac{2\pi n}{T}t\right)dt\right] \\
&= \frac{2}{T}\left[-H\frac{T}{2\pi n}\cos\left(\frac{2\pi n}{T}t\right)\Big|_0^W - L\frac{T}{2\pi n}\cos\left(\frac{2\pi n}{T}t\right)\Big|_W^T\right] \\
&= \frac{2}{T}\frac{T}{2\pi n}\left\{-H\left[\cos\left(\frac{2\pi n}{T}W\right)-\cos(0)\right]-L\left[\cos(2\pi n)-\cos\left(\frac{2\pi n}{T}W\right)\right]\right\} \\
&= \frac{1}{\pi n}\left[-H\cos\left(\frac{2\pi n}{T}W\right)+H-L+L\cos\left(\frac{2\pi n}{T}W\right)\right] \\
&= \frac{1}{\pi n}\left[(H-L)-(H-L)\cos\left(\frac{2\pi n}{T}W\right)\right] \\
&= \frac{H-L}{\pi n}\left[1-\cos\left(\frac{2\pi n}{T}W\right)\right] \\
&= \frac{H-L}{\pi n}\left[1-\cos(2\pi\cdot DC\cdot n)\right]
\end{aligned}
$$

Now let’s look at the magnitude and phase, which are given by:

$$c_n = \sqrt{a_n^2 + b_n^2}$$

$$\phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$$

In our case:

$$\begin{aligned} c_n &= \sqrt{\left(\frac{H-L}{\pi n}\right)^2 \sin^2\left(2\pi \cdot DC \cdot n\right) + \left(\frac{H-L}{\pi n}\right)^2 \left[1-\cos\left(2\pi \cdot DC \cdot n\right)\right]^2} \\ &= \frac{H-L}{\pi n}\sqrt{\sin^2\left(2\pi \cdot DC \cdot n\right) + \left[1-2\cos\left(2\pi \cdot DC \cdot n\right)+\cos^2\left(2\pi \cdot DC \cdot n\right)\right]} \\ &= \frac{H-L}{\pi n}\sqrt{2-2\cos\left(2\pi \cdot DC \cdot n\right)} \\ &= \frac{\sqrt{2}\left(H-L\right)}{\pi n}\sqrt{1-\cos\left(2\pi \cdot DC \cdot n\right)} \end{aligned} \tag{6}$$

and

$$\phi_n = \tan^{-1}\left(\frac{\frac{H-L}{\pi n}\left[1-\cos\left(2\pi \cdot DC \cdot n\right)\right]}{\frac{H-L}{\pi n}\sin\left(2\pi \cdot DC \cdot n\right)}\right) = \tan^{-1}\left(\frac{1-\cos\left(2\pi \cdot DC \cdot n\right)}{\sin\left(2\pi \cdot DC \cdot n\right)}\right) \tag{7}$$